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The exhaust trail from jet engines

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R.F. Jones

(Meteorological Office)

AIR MINISTRY

METEOROLOGICAL RESEARCH COMMITTEE

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1. Introduction

The "Mintra" line shown on the Meteorological Office Tephigram (Form 2810A) gives for any given height an indication of the temperatures above which contrails are unlikely to form. The theory leading to the determination of these temperatures is given in M.O.479(1). For many purposes it is more desirable, if possible, to forecast the temperatures below which contrails are likely to form. Using the Mintra temperatures for this purpose is unsatisfactory and a body of evidence has accumulated which suggests that, at least for jet engines, the Mintra temperature is too high for practical forecasting purposes, i.e. forecasts of the likelihood of contrails if the temperature falls below the Mintra value are not sufficiently precise and the majority of trails form first several degrees Centigrade below the Mintra value.

In this paper a somewhat different approach from that of M.O.479(1) is used. It assumes that for jet engines the temperature and issuing speed of the exhaust gases together with fuel consumption and mass flow through the engine are known, or can be determined, for various engines under different operating conditions.

2. Theory

The increased water content in the exhaust trail above that of the ambient air is produced by the burning of the fuel in the jet engine. For the kerosene fuel at present used 1 gm of such fuel when burnt yields 1.3 gm of water vapour. Knowing the rate of burning of the fuel and the mass flow of air through the engine it is possible, assuming all the fuel is burnt and all the water vapour produced is taken up by the air flowing through, to calculate the humidity mixing ratio of the exhaust air at the point where it is about to leave the jet pipe. Both the assumptions made above are reasonable since the oxygen supplied by the air flowing through the engine is very greatly in excess of that required for complete combustion of the fuel and the temperatures are such that it is impossible to visualise any loss of water vapour by condensation or other means.

If the mass flow of air through the engine is M lb/sec and the fuel consumed is F lb/sec the water vapour produced by the burning of the fuel will be WF lb/sec where W is the mass of water vapour produced by burning unit mass of fuel (W = 1.3 for kerosene). This water vapour will be included in the (M + F) lb/sec of exhaust gases along with the original water content of the M lb/sec of air. If x g/Kg is the mixing ratio of the ambient air, the mixing ratio x_e of the exhaust air will then be given by:-

$$x_e = \frac{1000\left(WF + \frac{Mx}{1000 + x}\right)}{(M + F) - \left(WF + \frac{Mx}{1000 + x}\right)} \text{ g/Kg}$$

$$= \frac{1000\left(WF + \frac{Mx}{1000 + x}\right)}{M\left(\frac{1000}{1000 + x}\right) - F(W - 1)} \text{ g/Kg} \qquad (1)$$

Because of the action of the compressors and the burning of the fuel the exhaust gas will arrive at the jet outlet at a temperature T and pressure P (different from the ambient temperature and pressure) and speed V and will immediately suffer adiabatic expansion to ambient pressure P_o. This will result in a cooling of the exhaust gas to a temperature T_1 given by:

$$T_1 = T\left(\frac{P_o}{P}\right)^{\frac{\gamma-1}{\gamma}} \qquad (2)$$

where γ is the ratio of the specific heats at constant pressure and constant volume for the mixture of gases comprising the exhaust gas.

This adiabatic cooling will leave the mixing ratio of the exhaust gas unchanged so that immediately after leaving the exhaust the gas is at a temperature T_1 given by equation (2) with a mixing ratio x_e given by equation (1). Further cooling of the air without condensation or mixing with the ambient air would leave the mixing ratio unchanged so that if mixing did not take place the exhaust air would become saturated when cooled to a temperature having a saturation mixing ratio of x_e.

The exhaust gases, however, leave the exhaust at a high speed v (of the order of 1,500 ft/sec relative to the aircraft and perhaps 800 ft/sec relative to the ambient air) and it is certain that considerable mixing with the ambient air will take place into such a jet.

As well as possessing thermal energy corresponding to the temperature T_1 the exhaust gases will also possess kinetic energy which will ultimately be given up to the trail in the form of heat as the speed of the exhaust jet is reduced by mixing with ambient air.

If 1 gm of exhaust gas moving with a speed v_1 cm/sec relative to the ambient air ($v_1 = v - V$, where V is the forward speed of the aircraft) is mixed with m gm of ambient air then considerations of momentum show that the speed v' of the (1 + m) gm of mixture will be given by

$$v' = \frac{v_1}{1 + m} \text{ cm/sec} .$$

The initial kinetic energy of 1 gm of exhaust gas is $\frac{1}{2} v_1^2$ erg and the final kinetic energy after mixing with m gm of ambient air is

$$\frac{1}{2}(1 + m)(v')^2 = \frac{v_1^2}{2(1 + m)} \text{ erg} .$$

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The loss of kinetic energy on mixing therefore, which appears in the mixture in the form of heat, is

$$\tfrac{1}{2} v_1^2 \left(1 - \frac{1}{1 + m}\right) = \frac{m v_1^2}{2(1 + m)} \text{ erg .}$$

The heat given to the $(1 + m)$ gm of mixture is therefore $\dfrac{m v_1^2}{2J(1 + m)}$ cal where J is the mechanical equivalent of heat ($= 4.18 \times 10^7$ erg/cal).

If c_p is the specific heat at constant pressure of the mixture of exhaust gas and ambient air the temperature increase of the mixture due to conversion of kinetic energy into heat will be

$$\tfrac{1}{2} \cdot \frac{v_1^2}{J c_p} \cdot \frac{m}{(1 + m)^2} \ ^\circ C \ .$$

In practice, since the mass flow of air through the jet engine is likely to be very much greater than the fuel consumption and since at heights important in the contrail problem the water content of the ambient air will be small, negligible error would be involved in taking the specific heat at constant pressure (c_p) of the mixture of exhaust gas and ambient air as identical with that of dry air (0.24 cal/gm/°C).

Considering now the thermal energies of the exhaust gas and ambient air we know that if 1 gm of gas of temperature T_1 °K is mixed at constant pressure with m gm of the same gas at temperature T_2 °K the temperature of the mixture will be $\dfrac{T_1 + mT_2}{1 + m}$ °K (see ref.(2), p.48).

Since, as we have seen above, the specific heats at constant pressure of exhaust gas and ambient air are unlikely to differ significantly from that of dry air the exhaust gas and ambient air can be considered to be of the same constitution for this purpose and the temperature of the mixture will be $\dfrac{T_1 + mT_2}{1 + m}$ °K where T_2°K is the temperature of the ambient air, when thermal energies alone are considered.

Taking conversion of kinetic energy also into account we find that the temperature T_m of the mixture of 1 gm of exhaust gas with m gm of ambient air will be given by:

$$T_m = \frac{T_1 + mT_2}{1 + m} + \tfrac{1}{2} \cdot \frac{v_1^2}{J c_p} \cdot \frac{m}{(1 + m)^2} \qquad (3)$$

The mixing ratio, x_m, of the mixture of 1 gm of exhaust gas (mixing ratio x_e) with m gm of ambient air (mixing ratio x) will be:

$$x_m = \frac{1000\phi}{1 + m - \phi} \text{ g/Kg}$$

$$\text{where } \phi = \frac{x_e}{1000 + x_e} + \frac{mx}{1000 + x}$$

Using equation (1) it can be shown that

$$x_m = \frac{1000\{1000WF + x[(1 + m)M + (W + m)F]\}}{1000(1 + m)(M + F) - F[1000W + x(W - 1)]} \qquad (4)$$

Now x in the contrail problem is unlikely greatly to exceed 1g/Kg and W (water vapour produced by burning 1 gm of fuel) is of the order of 1.3 so that x(W-1) in the denominator of (4) is small compared with 1000W and without important error we can write:-

$$x_m = \frac{1000WF + x[(1 + m)M + (W + m)F]}{(1 + m)(M + F) - WF}$$

$$= \frac{1000WF}{(1 + m)(M + F) - WF} + x\left[1 + \frac{F(2W - 1)}{(1 + m)(M + F) - FW}\right]$$

$$= 1000W\beta + x\left[1 + (2W - 1)\beta\right] \text{ where } \beta = \frac{F}{(1 + m)(M + F) - WF}$$

$$= \beta\left[1000W + (2W - 1)x\right] + x \qquad (5)$$

and since (2W-1)x is also small compared with 1000W and β itself is usually small especially when m (the degree of mixing) becomes large it is often legitimate to make the further approximation

$$x_m = 1000W\beta + x \qquad (6)$$

For constant m, β is a function only of the engine performance and W depends only on the type of fuel used so that the expression (6) conveniently separates the effects of engine performance and fuel type from the properties of the ambient air.

Equations (3) and (6) (or more strictly (4)), therefore, give the temperature and mixing ratio respectively of the gases in the exhaust trail as long as the air remains unsaturated. Thus, knowing the engine performance details, it is possible to obtain values of T_m, x_m for various conditions of ambient temperature and mixing ratio and different degrees of mixing with the ambient air. The values of T_m can conveniently be converted by means of meteorological tables into values of the mixing ratio for which at the ambient pressure concerned the air would be saturated with respect to ice (or water). Two sets of curves can then be drawn on the same diagram showing the variation with m of saturation mixing ratio in the trail and x_m for various values of the ambient temperature and mixing ratio. Where the curves intersect will give the degree of mixing which needs to be achieved before the exhaust trail becomes saturated and the saturation mixing ratio within the trail at which this occurs. In particular if the ambient air is saturated with respect to ice (or water) critical temperatures will be defined above which saturation in the trail with respect to ice (or water) can never be achieved. These temperatures would define the "Mintra" temperature for the level concerned. Similarly if the ambient air is completely dry (x = o) "Drytra" temperatures would also be defined.

As an example of the use of such a diagram let us assume reasonable values for the various parameters involved for a jet engine operating at an ambient pressure of 200 mb.

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Such values may be: $T_1 = 600°A$, $v_1 = 2.5 \times 10^4$ cm/sec, $W = 1.3$, $F = 0.5$, $M = 32.5$ together with known values of J and c_p ($J = 4.18 \times 10^7$ erg/cal, $c_p = 0.24$ cal gm^{-1} $°C^{-1}$).

It can then be shown using equations (3) and (6) (in accurate computations equation (4) can be used) that T_m, x_m are given by:

$$T_m = \frac{600 + 631\,m}{(1+m)^2} + T_2\left[1 - \frac{1}{1+m}\right]$$

$$x_m = \frac{650}{33m + 32.35} + x$$

where T_2 and x are the temperature and mixing ratio respectively of the ambient air.

Values of T_m for various values of T_2 and m and of x_m for various values of m assuming $x = 0$ (completely dry ambient air) are shown in Table I. Also shown are saturation mixing ratios over ice and water, x_i, x_w corresponding to T_m at a pressure of 200 mb; for convenience in the use of the Meteorological Tables(3) degrees Centigrade + 273 have been used in preference to °K.

TABLE I

Temperature T_m and mixing ratio x_m in the exhaust trail for various degrees of mixing with completely dry ambient air at temperature T_2. (x_i is the saturation mixing ratio over ice at 200 mb corresponding to T_m, x_w that over water)

m	9	19	29	39	49	59	99	199	499	999
					$T_2 = 210$ °A					
T_m	251.8	231.0	224.0	220.5	218.4	217.0	214.2	212.1	210.8	210.4
x_i	2.87	0.318	0.138	0.089	0.069	0.057	0.040	0.030	0.025	0.024
x_w	3.54	0.478	0.222							
					$T_2 = 220°A$					
T_m	260.8	240.5	233.7	230.3	228.2	226.8	224.1	222.1	220.8	220.4
x_i	6.68	0.912	0.434	0.294	0.230	0.195	0.140	0.109	0.092	0.088
x_w	7.57	1.251	0.634	0.444	0.354	0.304	0.225	0.178		
					$T_2 = 225°A$					
T_m	265.3	245.2	238.5	235.1	233.1	231.8	229.1	227.0	225.8	225.4
x_i	10.02	1.489	0.736	0.506	0.405	0.349	0.255	0.199	0.173	0.165
x_w	10.86	1.953	1.028	0.732	0.595	0.520	0.390	0.310	0.272	0.260
					$T_2 = 230°A$					
T_m	269.8	250.0	243.3	240.0	238.0	236.7	234.0	232.0	230.8	230.4
x_i	14.8	2.406	1.220	0.864	0.696	0.604	0.448	0.357	0.311	0.298
x_w	15.4	3.019	1.635	1.191	0.979	0.861	0.653	0.531	0.469	0.449
					$T_2 = 240°A$					
T_m	278.8	259.5	253.0	249.8	247.8	246.5	243.9	241.9	240.8	240.4
x_i	30.1	5.95	3.224	2.361	1.935	1.699	1.300	1.056	0.941	0.902
x_w	30.1	6.81	3.929	2.970	2.482	2.202	1.729	1.431	1.288	1.239
x_m	1.973	0.986	0.657	0.493	0.394	0.328	0.197	0.099	0.039	0.020

/These

These values are shown plotted in Fig. 1. Curves allowing for the mixing ratio, x, of the ambient air can readily be plotted since all that is required (equation (6)) is to add the value of x at each stage to the corresponding value for a completely dry atmosphere. Two such curves are plotted in Fig. 1 corresponding to values of x representing saturation of the ambient air at temperatures of 230°A and 240°A (x = 0.284 and 0.864 g/Kg respectively).

If the assumed values of the parameters were accurate values for a particular jet engine then for this engine the following deductions could be made from Fig. 1 for the pressure level of 200 mb:

(i) If the ambient air is completely dry saturation of the exhaust trail with respect to ice would just be achieved if the ambient temperature was 225°A but for ambient temperatures higher than this value saturation would never be achieved. 225°A would, therefore, be the "Drytra" temperature for this engine at a pressure of 200 mb. If saturation with respect to water is required the corresponding "Drytra" temperature would be about 221°A.

(ii) At an ambient temperature of 225°A saturation of the exhaust trail with respect to ice is just achieved after mixing with 39 times its own weight of completely dry ambient air and, by reference to Table I, at a temperature of the trail of 235.1°A.

(iii) At an ambient temperature of 220°A saturation of the trail with respect to ice would be achieved after mixing with about 19 times its own weight of dry ambient air and saturation would be maintained until mixed with about 160 times its own weight of ambient air. The maximum free ice content of the air would be about 0.22 g/Kg when mixed with about 29 times its own weight of dry ambient air. The trail would become unsaturated again at a temperature still some 3°C higher than that of the ambient air.

(iv) If the ambient air is saturated with respect to ice at 230°A the trail will also become saturated with respect to ice after mixing with about 50 times its own weight of ambient air and thereafter remain saturated but if the ambient air is saturated at 240°A the trail will never become saturated. The "Mintra" temperature for this engine at 200 mb pressure lies therefore between 230°A and 240°A and its accurate value can readily be determined.

(v) Similar deductions can be made for varying values of humidity of the ambient air by suitable adjustment of the curves for the appropriate values of x.

(vi) For low ambient temperatures (e.g. below 210°A) a saturated trail is likely to be attained with a low degree of mixing whatever the ambient humidity and the trail will remain saturated until very greatly mixed with ambient air. The effect of ambient humidity is likely therefore to become less marked as the ambient temperature decreases.

3. The effect of release of latent heat of condensation or sublimation

The argument of Section 2 is valid as long as the gases in the exhaust trail remain unsaturated but as soon as saturation of the trail is achieved a further source of heat becomes operative, namely the release of the latent heat of condensation or sublimation once water drops or ice crystals begin to form in the trail. There is some evidence to support the belief that even at temperatures well below the freezing point direct sublimation from the vapour in the form of ice crystals does not take place as soon as supersaturation with respect to ice is achieved but that saturation with respect

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to water must first be achieved and the initial deposition of surplus water vapour is in the form of supercooled water droplets. At very low temperatures such as commonly occur in contrail formation, of the order of 230°K and below, subsequent freezing of these droplets is likely to be very rapid.

Suppose condensation in the liquid form has occurred at a temperature, T_m, of about 230°K until 0.01 g/Kg of liquid water has been condensed. The condensation of this quantity of liquid from the vapour will release 0.01 x 620 cal/Kg sufficient to raise the temperature of the air by $\frac{0.01 \times 620}{0.24 \times 1000} = 0.03°C$, assuming the specific heat of the exhaust gases to be the same as that of dry air. If we suppose that this water now freezes the latent heat liberated (0.01 x 56 cal/Kg) will have an insignificant effect on the temperature but, with ice crystals now present, direct sublimation from the vapour onto the ice crystals will proceed rapidly until the vapour content is reduced to that corresponding to saturation with respect to ice. Since the difference in saturation mixing ratios with respect to water and ice at about 230°K is about 0.15 g/Kg this amount of water vapour must be converted directly into ice releasing 0.15 x 676 cal/Kg.

If x_i is the saturation mixing ratio with respect to ice, at temperature T_m, therefore, the total latent heat contributed to the trail would be $(x_m - x_i)$ x 676 cal/Kg which in the case discussed would be sufficient to raise the temperature of the gas by about 0.5°C. This full rise of temperature will not be achieved since as the temperature rises the saturation mixing ratio will also rise (by about 0.02 g/Kg for a temperature rise of 0.5°C from about 234°K) so that a balance is finally achieved whereby the final temperature rise due to release of latent heat is about 0.4°C and the final solid ice content in the trail is somewhat less than $(x_m - x_i)$g/Kg. As mixing with the ambient air proceeds a stage will be reached when the solid ice content $(x_m - x_i)$ in the trail will reach a maximum and the temperature increase due to latent heat will also be a maximum. Thereafter $(x_m - x_i)$ will decrease as some of the solid content is evaporated in order to bring the entrained air up to saturation with respect to ice at the temperature of the trail. The gain of latent heat in the early stages of mixing will then be balanced in the later stages by loss of heat to evaporate the ice until when all the ice has been evaporated the temperature T_m and mixing ratio x_m will again be those determined by equations (3) and (4). Thus, in the stage when the exhaust trail contains ice crystals the temperature T_m will be greater than that given by equation (3) by an amount given approximately by $(x_m - x_i) \times \frac{676°C}{240}$ and the true ice content will be slightly less than that deduced using x_m from equation (4).

Instead of making corrections to T_m and $(x_m - x_i)$ it is more convenient to deal with T_m and x_m as determined by equations (3) and (4) and refer to $(x_m - x_i)$ as the "effective ice content" regarding a small proportion of $(x_m - x_i)$ as being earmarked to provide the water vapour required to maintain saturation as T_m rises on account of the release of latent heat of sublimation of the remainder of $(x_m - x_i)$.

4. The visible exhaust trail (contrail)

The argument of Section 2 enables a deduction to be made of when, if at all, the exhaust trail is saturated with respect to ice or water. Saturation alone, however, is not sufficient to make the trail visible, there must be condensation to liquid water or sublimation to ice to such an extent that the number and size of particles render the trail visible. What this number and size must be has never been clearly demonstrated and in any case must depend on the conditions of viewing, e.g., the distance away and contrast with the background as well as the acuity of the observer's eye.

/Appelman(4)

Appelman[4] has considered as necessary a minimum water (or ice) content of 0.004 g/m^3 for a faint trail and 0.01 g/m^3 for a distinct trail but recognises that these definitions are quite arbitrary.

From observations of the level of first formations of contrails from an instrumented aircraft having engines of known characteristics it is possible by the use of a diagram such as Fig.1 to make a close estimation of the excess ice crystal content in the trail. For example, if an aircraft using the hypothetical engine discussed in Section 2 first made a visible trail at 200 mb when the ambient temperature was 220°A and the mixing ratio 0.01 g/Kg (equivalent to a relative humidity with respect to ice of 12%) Fig.1 would show the maximum effective ice crystal content of the trail to be 0.21 g/Kg. Since the temperature in the trail at this point would be 233.7°A (Table I) the maximum effective ice crystal content would be $\frac{0.21}{0.298}$ g/m^3 = 0.07 g/m^3, the density of air at 233.7°A and 200 mb pressure being 0.298Kg/m^3. Observations such as this have suggested that the minimum effective ice crystal content necessary to make the trail visible is much in excess of the value of 0.004 g/m^3 quoted by Appelman. It is worth noting, however, that if it is considered that condensation to water droplets is first necessary in the trail followed by rapid freezing of these droplets then the ice crystal content of the trail, when freezing and reduction of the mixing ratio to saturation with respect to ice has been completed, will certainly be greater than the Appelman criterion. Fig.1 shows that in the example considered saturation with respect to water is achieved for the same degree of mixing as gives the maximum ice crystal content, so that the requirement for saturation with respect to water automatically leads to a maximum effective ice crystal content in the trail of 0.07 g/m^3. If, however, the minimum ice crystal content to make the trail visible is in excess of 0.07 g/m^3 the critical temperature for visible trail formation becomes less than that given for saturation with respect to water. If, for example, the minimum effective ice crystal content is 0.1 g/m^3 (equivalent to about 0.33 g/Kg at the probable trail temperature at 200 mb pressure) Fig.1 shows that for completely dry ambient air the critical temperature for visible trail formation is about 218°A compared with 221°A if saturation with respect to water is used as the criterion and 225°A for saturation with respect to ice. It is therefore clearly desirable to determine as accurately as possible the minimum ice crystal content necessary to make the trail visible and any variation with height. Having determined this value it is a simple matter to adjust the ambient mixing ratio curves of diagrams like Fig.1, taking account of the variation of density with temperatures in the trail for various ambient temperatures, to ensure that the minimum visibility criterion is met, leading to new values of "Mintra" and "Drytra" temperatures. These new values will not differ from those determined assuming initial saturation of the trail with respect to water unless (a) the trail remains as a water cloud for an appreciable time after formation, or (b) if freezing of the water droplets in the trail is rapid, the minimum ice crystal content necessary to make the trail visible is greater than would result from the change from saturation with respect to water to saturation with respect to ice. A further point requiring examination, therefore, is how quickly the change of state within the trail takes place.

5. The persistence of contrails

The persistence of a trail once visible must clearly depend on the degree of mixing with the ambient air before the minimum ice crystal content for visibility is again reached and the time taken for this mixing to be achieved. Observations of the lengths of trails from an instrumented aircraft having engines of known characteristics can again be used in association with diagrams of the form of Fig.1 to assist in the forecasting of the length of the visible trail. For example, using Fig.1, we see that saturation with respect to water is achieved at an ambient air temperature of 220°A if the air is completely dry after mixing corresponding to m = 25 and saturation is again reached for m = 71; if freezing has taken place there will be an ice crystal content in

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the exhaust trail until saturation with respect to ice is once more achieved at m = 166. The visible trail will certainly vanish therefore before m = 166 and according to the value of minimum ice content necessary for visibility a value of m can be found which will mark the beginning and end of the visible trail. Aircraft observations of the distances from the aircraft of commencement and cessation of trails will therefore give, by use of diagrams such as Fig.1, a mean value for the way m varies with distance, making possible an assessment of the length of trails for that particular aircraft for any occasion when the ambient temperature and humidity are known. It is clear from the form of the curves in Fig.1 that the effect of ambient humidity may be considerable on the length of a trail, e.g., if the ambient air is saturated at 220°A (mixing ratio 0.084 gm/Kg) saturation with respect to ice is not reached again until m exceeds 1000.

By determining a minimum value of m corresponding to "persistent" trails it becomes possible to forecast for a particular aircraft whose engine characteristics are known temperatures at a certain level analogous to "Drytra" and "Mintra" temperatures below which and above which persistent trails are likely and not likely to form respectively.

6. The diameter of the exhaust trail

Suppose the exhaust gas immediately before leaving the exhaust pipe is at temperature T, pressure P and moving at speed v relative to the ambient air. After adiabatic expansion to ambient pressure P_o, reducing T to T_1, and mixing with m times its own weight of ambient air of temperature T_2, it will have a temperature of T_m, pressure P_o and speed v_m where

$$v_m = \frac{v}{1 + m}$$

and

$$T_m = \frac{T_1 + mT_2}{1 + m} + \tfrac{1}{2} \frac{v^2}{Jc_p} \frac{m}{(1 + m)^2} \qquad \text{(equation 3)}$$

The total transfer of mass per second across the whole cross-section of the trail is constant so that if ρ, ρ_m are the densities of the exhaust gas before and after mixing, A the cross-sectional area of the jet pipe and A_m the cross-sectional area of the trail after mixing

$$\rho . A . v = \rho_m \cdot A_m \cdot v_m$$

$$= \rho_m \cdot A_m \cdot \frac{v}{1 + m} \quad \text{i.e.} \quad \rho . A = \frac{1}{1 + m} \cdot \rho_m \cdot A_m .$$

$$\text{Now } \rho = \frac{P}{RT}, \quad \rho_m = \frac{P_o}{RT_m}$$

$$\text{so that } \frac{P}{RT} \cdot A = \frac{1}{1 + m} \cdot \frac{P_o}{RT_m} \cdot A_m$$

$$\text{giving } \frac{A_m}{A} = (1 + m) \cdot \frac{T_m}{T} \cdot \frac{P}{P_o}$$

/Since

Since T_1 is the temperature of the exhaust gas after adiabatic expansion from pressure P to pressure P_o we have $\frac{P}{P_o} = \left(\frac{T}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$ and substituting for T_m and $\frac{P}{P_o}$ we have

$$\frac{A_m}{A} = \left\{ T_1 + mT_2 + \tfrac{1}{2} \cdot \frac{v^2}{J.c_p} \cdot \frac{m}{1+m} \right\} \left(\frac{T}{T_1}\right)^{\frac{1}{\gamma-1}}.$$

For the hypothetical engine discussed in Section 2 we then have, assuming $T = 700°K$,

$$\frac{A_m}{A} = 0{\cdot}0026 \left\{ 600 + mT_2 + \frac{31m}{1+m} \right\}$$

$$\doteq 1{\cdot}64 + {\cdot}0026 mT_2 \quad \text{since} \quad \frac{m}{1+m} \doteq 1$$

for all but small values of m.

If D_m is the diameter of the cross-section of the trail and D the diameter of the jet pipe exit, both assumed circular, we have

$$\frac{D_m}{D} = \sqrt{\frac{A_m}{A}}$$

$$= \sqrt{1{\cdot}64 + {\cdot}0026\ mT_2}$$

Values of $\frac{D_m}{D}$ for various values of m at an ambient temperature of 200°K are shown in Table II.

Table II

Change in trail diameter with varying entrainment of ambient air at a temperature 200°K

m	D_m/D
10	2.61
20	3.47
30	4.15
40	4.74
50	5.26
60	5.73
80	6.58
100	7.32
1000	22.8

It is clear from the form of Fig.1 that saturation of the exhaust trail will never be achieved until mixing of the order of m = 10 or more has been achieved so that on first becoming visible the contrail diameter is likely to be about 3 times the diameter of the jet orifice. The diameter of the trail

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will increase thereafter as the mixing with the ambient air proceeds but, unless the mixing is very rapid, the rate of change of diameter will be small. The observations suggested in Section 5 will give the variation of m with distance for a particular aircraft but it may be noted that observations from the ground of persistent trails suggest usually a small rate of growth of the diameter of the trail with distance although the trail starts quite close to the aircraft. This suggests an initial high rate of entrainment into the exhaust trail followed by relatively slow entrainment. This is shown also by the type of persistent trail formed by an aircraft having four jet engines. It is seen that the four trails from the engines very quickly merge into two as the trails from neighbouring engines on each side of the aircraft amalgamate. The distance between neighbouring jet pipes on the same side of the aircraft is small and, as Table II suggests, each trail will very quickly grow to more than three times the diameter of the jet pipe so that amalgamation of neighbouring trails is likely to be rapid. The distance apart of the pairs of engines, however, separated as they are by the fuselage, will be many diameters of the jet pipe and Table II suggests that the two remaining trails are unlikely to merge into one until m exceeds perhaps 100 or more. The very great distance behind the aircraft at which the two trails merge into one indicates that although mixing corresponding to m = 10 is very rapid that corresponding to m = 100 may not take place until the trail is several miles long.

Summary

Knowing certain performance data of a jet engine it is possible by a simple mixing theory to deduce the temperatures and humidity mixing ratio distribution in the exhaust trail. In particular "Mintra" and "Drytra" temperatures can be determined. By comparison with contrail observations from an instrumented aircraft having engines of known characteristics it will be possible to determine the minimum water or ice content in the trail necessary to make the trail visible, leading to the possibility of forecasting the length, diameter and persistence of visible trails.

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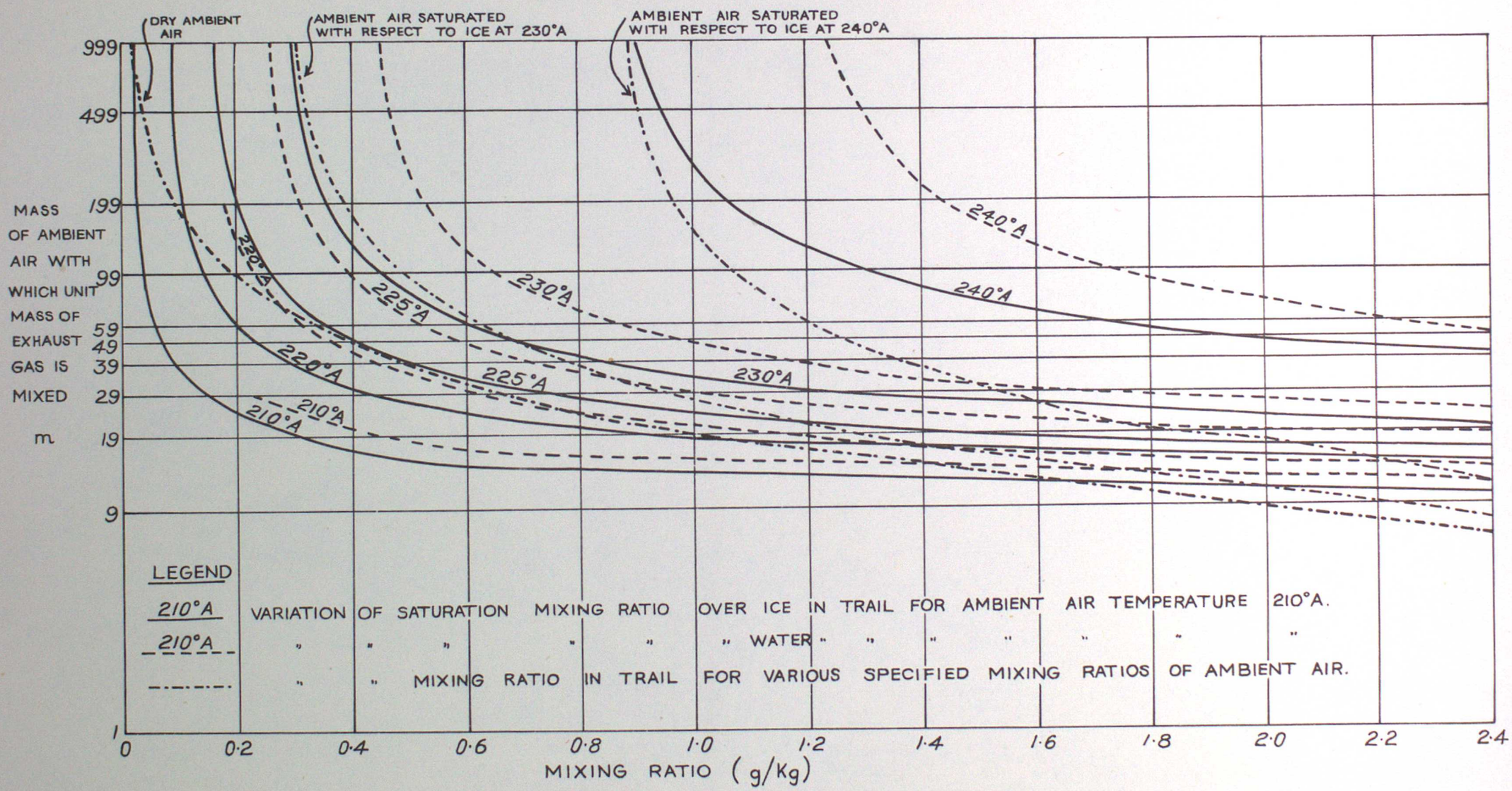

FIG. 1. CONDITION IN EXHAUST TRAIL AT 200mb.